

FUNCTIONS OF BOUNDED RADIUS ROTATION OF ANALYTIC FUNCTIONS

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ABSTRACT

The classes $V_k(\alpha)$, $R_k(\alpha)$, $V_{k1}(\alpha)$ and $R_{k1}(\alpha)$ of analytic function, We establish a relation between the functions of bounded boundary and bounded radius rotations.

KEYWORDS: Analytic, Analytic Functions with Bounded Radius and Boundary Rotations, Starlike, Convex, Subordination

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1. INTRODUCTION

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

Which are analytic in the unit disc $E = \{z: |z| < 1\}$.

The definition of subordination is $f \in A$ is subordinate to $g \in A$, written as $f \prec g$, there exists Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in E$) such that $f(z) = g(w(z))$, In particular, when g is univalent, then the above subordination is equivalent to $f(0) = g(0)$ and $f(E) \subseteq g(E)$, for any two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in E). \quad (1.2)$$

then convolution

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in E) \quad (1.3)$$

We denote by $s^*(\alpha)$, $c(\alpha)$, ($0 \leq \alpha < 1$) the classes of starlike and convex functions of order α ,

respectively defined by

$$\begin{aligned} s^*(\alpha) &= \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, z \in E \right\} \\ c(\alpha) &= \left\{ f \in A; zf'(z) \in s^*(\alpha), z \in E \right\} \end{aligned} \quad (1.4)$$

For $\alpha = 0$, we have the well know classes of starlike and convex univalent functions denoted s^* and c , respectively.

Let $P_k(\alpha)$ be the class of functions $P(z)$ analytic in the unit disc E satisfying the properties $P(0)=1$ and

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{P(z) - \alpha}{1 - \alpha} \right| d\theta \leq k\pi, \quad (1.5)$$

Where $z = re^{i\theta}$, $k \geq 2$, and $0 \leq \alpha < 1$, using Herglotz – Stieltjes formula

$$P(z) = \left(\frac{k}{4} + \frac{1}{2} \right) P_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) P_2(z), \quad z \in E \quad (1.6)$$

Where $P(\alpha)$ is the class of functions with real part greater then α and $P_i \in P(\alpha)$, for $i = 1, 2$ we define the following classes

$$\begin{aligned} R_k(\alpha) &= \left\{ f : f \in A \text{ and } \frac{[\lambda z^2 f''(z) + zf'(z)]}{\lambda z(f'(z) - f'(-z)) + (1-\lambda)(f(z) - f(-z))} \in P_k(\alpha), 0 \leq \lambda < 1 \right\}, t > 0 \\ V_k(\alpha) &= \left\{ f : f \in A \text{ and } \frac{[\lambda z^2 f''(z) + zf'(z)]}{\lambda z(f''(z) - f''(-z)) + [(1-\lambda)(f'(z) - f'(-z))]} \in P_k(\alpha), 0 \leq \lambda < 1 \right\}, t > 0 \end{aligned} \quad (1.7)$$

$$R_{k1}(\alpha) = \left\{ f : f \in A \text{ and } \frac{[\gamma \lambda z^3 f'''(z) + (2\gamma\lambda + \gamma + \lambda)z^2 f''(z) + zf'(z)]}{\gamma \lambda z^2 (f''(z) - f''(-z)) + (\lambda - \gamma)z(f'(z) - f'(-z)) + (1 - \lambda + \gamma)(f(z) - f(-z))} \in P_k(\alpha), \right. \\ \left. 0 \leq \lambda \leq \gamma < 1. \right\} \quad (1.8)$$

$$V_{k1}(\alpha) = \left\{ f : f \in A \text{ and } \frac{[\gamma \lambda z^3 f'''(z) + (2\gamma\lambda + \gamma + \lambda)z^2 f''(z) + zf'(z)]}{\gamma \lambda z^2 (f''(z) - f''(-z)) + (\lambda - \gamma)z(f'(z) - f'(-z)) + [(1 - \lambda + \gamma)(f(z) - f(-z))]} \in P_k(\alpha), \right. \\ \left. 0 \leq \lambda \leq \gamma < 1. \right\} \quad (1.9)$$

We note that

$$f \in V_k(\lambda) \Leftrightarrow zf' \in R_k(\lambda) \quad f \in V_{k1}(\lambda) \Leftrightarrow zf' \in R_{k1}(\lambda) \quad (1.10)$$

For $\lambda = 0, t = 0, \gamma = 0$ we obtain the well known classes R_k, R_{k1}, V_k, V_{k1} of analytic functions with bounded radius and bounded boundary rotations, respectively. These classes are studied by Noor [3] in more details, also it can easily be seen $R_k^2(\alpha) = S^*(\alpha)$ and $V_2(\alpha) = c(\alpha)$. Goel [6] proved that $f \in c(\alpha)$ implies that $f \in S^*(\alpha)$ and $f \in S^*(\alpha)$

$$\text{Where } \beta = \beta(\alpha) = \begin{cases} \frac{4^\alpha(1-2\alpha)}{4-2^{2\alpha+1}}, & \alpha \neq \frac{1}{2} \\ \frac{1}{2} \ln 2 & , \alpha = \frac{1}{2} \end{cases}$$

and the result is sharp. In this paper, we prove the result of Goel [6] for the classes $V_k(\alpha), R_k(\alpha), V_{k1}(\alpha)$ and $R_{k1}(\alpha)$ by using convolution and Subordination techniques.

2. PRELIMINARY RESULTS

We need the following results to obtain our results.

Lemma 2.1:

Let $u = u_1 + iu_2, v = v_1 + iv_2$ and $\Psi(u, v)$ be a complex valued function satisfying the conditions

(i) $\Psi(u, v)$ is continuous in a domain $D \subset C^2$.

(ii) $(1,0) \in D$ and $\text{Re}\psi(1,0) > 0$, when ever $(iu_2, v_1) \leq 0$ when ever $(iu_2, v_1) \in D$ and $v_1 \leq -\left(\frac{1}{2}\right)(1+u_2^2)$

If $h(z) = 1 + c_1z + \dots$ is a function analytic in E . Such that $(h(z), zh'(z)) \in D$ and $\text{Re}\psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\text{Re } h(z) > 0$ in E .

3. MAIN RESULTS

Theorem 3.1

Let $f \in V_k(\alpha)$, then $f \in R_k(\alpha)$, where

$$\beta = \frac{1}{4}(2\alpha - (1 + \lambda) + 2\alpha\lambda) + \sqrt{\frac{(1 + \lambda)^2 + 4\alpha^2 + 4\alpha^2\lambda^2 - 4\alpha(1 + \lambda) + 8\alpha^2\lambda - 4\alpha(1 + \lambda)}{+8 - 8\lambda + 8\lambda + (1 + \lambda^2)}}$$

Proof:

Let

$$\frac{(\lambda z^2 f''(z) + z f'(z))}{\lambda z(f(z) - f(-z)) + (1 - \lambda)(f(z) - f(-z))} = (1 - \beta)P(z) + \beta \tag{3.1}$$

$$= (1-\beta) \left[\left(\frac{k}{4} + \frac{1}{2} \right) P_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) P_2(z) \right] + \beta \quad (3.2)$$

$P(z)$ is analytic in E with $P(0)=1$ then

$$\frac{[\lambda z^2 f''(z) + z f'(z)]}{\lambda z [f'(z) - f'(-z)] + (1-\lambda)[f(z) - f(-z)]} = (1-\beta)p(z) + \beta \quad (3.3)$$

$$+ \frac{[\lambda z^2 f''(z) + z f'(z) + (1-\beta)p'(z)]}{\lambda z [f'(z) - f'(-z)] + (1-\lambda)(1-\beta)p(z) + \beta}$$

$$\frac{1}{(1-\alpha)} \left[\frac{[\lambda z^2 f''(z) + z f'(z)]}{\lambda z [f'(z) - f'(-z)] + (1-\lambda)[f(z) - f(-z)]} - \alpha \right]$$

$$= \frac{1}{(1-\alpha)} \left[(1-\beta)p(z) + \beta - \alpha + \frac{[\lambda z^2 f''(z) + z \lambda f'(z) + (1-\lambda)z](1-\beta)p'(z)}{\lambda z [f'(z) - f'(-z)] + (1-\lambda)[(1-\beta)p(z) + \beta]} \right] \quad (3.4)$$

$$= \frac{\beta - \alpha}{1-\alpha} + \frac{1-\beta}{1-\alpha} \left[p(z) + \frac{\left(\frac{1}{1-\beta} \right) (L+M)p'(z)}{(N+Q)\left(p(z) + \frac{\beta}{1-\beta} \right)} \right]$$

Where

$$L = \lambda z^2 f''(z)$$

$$M = \lambda z f'(z) + (1-\lambda)z$$

$$N = \lambda z [f'(z) - f'(tz)]$$

$$Q = (1-\lambda)$$

Since $f \in V_k(\alpha)$, it implies that

$$= \frac{\beta - \alpha}{1-\alpha} + \frac{1-\beta}{1-\alpha} \left[p(z) + \frac{\left(\frac{1}{1-\beta} \right) (L+M)p'(z)}{(N+Q)\left(p(z) + \frac{\beta}{1-\beta} \right)} \right] \in p_k, \quad z \in E \quad (3.5)$$

We define

$$\varphi_{a,b}(z) = \frac{1}{1+b} \frac{z}{(1-z)\alpha} + \frac{b}{1+b} \frac{z}{(1-z)^{1+a}} \quad (3.6)$$

with $a = \frac{1}{1-\beta}$, $b = \frac{\beta}{1-\beta}$ by using (3.1) with convolution, see [5], we have, that

$$\frac{\varphi_{a,b}(z)}{z} * p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \left[\frac{\varphi_{a,b}(z)}{z} * p_1(z) \right] - \left(\frac{k}{4} - \frac{1}{2}\right) \left[\frac{\varphi_{a,b}(z)}{z} * p_2(z) \right]$$

Implies

$$p(z) + \frac{a(L+M)p'(z)}{(N+Q-R)(p(z)+b)} = \left(\frac{k}{4} + \frac{1}{2}\right) \left(p_1(z) + \frac{a(L+M)p_1'(z)}{(N+Q)(p_1(z)+b)} \right) - \left(\frac{k}{4} + \frac{1}{2}\right) \left(p_2(z) + \frac{a(L+M)p_2'(z)}{(N+Q)(p_2(z)+b)} \right) \tag{3.7}$$

Thus from (3.5) and (3.7) we have

$$\frac{\beta - \alpha}{1 - \alpha} + \frac{(1 - \beta)}{(1 - \alpha)} \left[p_i(z) + \frac{a(L+M)p_i'(z)}{(N+Q)(p_i(z)+b)} \right] \in p \quad i=1,2 \tag{3.8}$$

We now from functional $\psi(u, v)$ by choosing $u = p_i(z)$, $v = zp_i'(z)$ in (3.8) and note that the first two conditions of Lemma 2.1 are likely satisfied, we check the condition as follows

$$\begin{aligned} \operatorname{Re}[\psi(iu_2, v_1)] &= \frac{1}{1-\alpha} \left[(\beta - \alpha) + \operatorname{Re} \left(\frac{\lambda z^2 f''(z)p_i'(z) + f'(z)v_1}{\left(iu_2 + \frac{\beta}{1-\beta}\right) [\lambda z(f''(z) - f''(tz)) + (1-\lambda)[f'(z) - f'(z)']]} \right) \right] \\ &= \frac{1}{1-\alpha} \left[(\beta - \alpha) + \frac{\left(\lambda z^2 f''(z)p_i'(z) + f'(z)v_1\right) \left(\frac{\beta}{1-\beta}\right)}{\left(iu_2 + \frac{\beta}{1-\beta}\right) \left(\lambda z(f''(z) + f'(z)) + (1-\lambda)\right)} \right] \\ &\leq \frac{1}{1-\alpha} \left[(\beta - \alpha) - \frac{\frac{1}{2}((1+u_2^2)f'(z) + \lambda z^2 f''(z)) \left(\frac{\beta}{1-\beta}\right)}{\left(u_2^2 + \left(\frac{\beta}{1-\beta}\right)^2\right) \left((f(z) - f(z)) + \lambda z^2 f''(z)\right)} \right] \\ &= \frac{2(\beta - \alpha) - ((1+u_2^2)f'(z) + \lambda z^2 f''(z)) \left(\frac{\beta}{1-\beta}\right)}{2\left(u_2^2 + \left(\frac{\beta}{1-\beta}\right)^2\right) \left((f(z) - f(z)) + \lambda z^2 f''(z)\right)} \tag{3.9} \\ &= \frac{A + Bu_2^2}{2c}, \quad 2C > 0, \end{aligned}$$

Where

$$\begin{aligned}
 A &= \frac{\beta}{(1-\beta)^2} \left[2(\beta-\alpha)\beta[(f(z)-f'(z)) + \lambda z^2 f''(z)]' - [f'(z) + \lambda z^2 f''(z)](1-\beta) \right] \\
 B &= \frac{1}{(1-\beta)} \left[2(\beta-\alpha)(1-\beta) [(f(z)-f'(z)) + \lambda z^2 f''(z)]'' - \beta [f'(z) + \lambda z^2 f''(z)] \right] \quad (3.10) \\
 C &= \left[u_2^2 + \left(\frac{\beta}{1-\beta} \right)^2 \right] (1-\alpha) [(f(z)-f'(z)) + (1+\lambda)\lambda z^2 f''(z)]'
 \end{aligned}$$

The right hand side of (3.9) is negative if $A \leq 0$ we have

$$\beta = \beta(\alpha) = \frac{1}{4}(2\alpha - (1+\lambda) + 2\alpha\lambda) + \sqrt{\frac{(1+\lambda)^2 + 4\alpha^2 + 4\alpha^2\lambda^2 - 4\alpha(1+\lambda) + 8\alpha^2\lambda - 4\alpha(1+\lambda)}{+8 - 8\lambda + 8\lambda + (1+\lambda^2)}}$$

Theorem 3.2

Let $f \in V_{k1}(\alpha)$, then $f \in R_{k1}(\alpha)$, where

$$\beta = \frac{1}{4}(2\alpha - (1+\lambda)(1+\gamma) + 2\alpha\gamma\lambda) + \sqrt{\frac{(1+\lambda+\gamma)^2 + 4\alpha^2\lambda^2\gamma + 6\alpha^2\gamma^2\lambda^2 - 4\alpha(1+\gamma)(1+\lambda)}{+16\alpha^2\lambda\gamma - 4\alpha(1+\lambda)(1+\gamma) + 8(1+\lambda)(1+\gamma) - 12\gamma\lambda}}$$

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